Using the relation between a plane projectivity and the fundamental matrix

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Abstract

This paper explores the relation between the projectivity associated with the motion of a plane and the *fundamental matrix*. It is straight forward to test for the existence of and stably estimate the projectivity $\mathcal{P}$ of a moving plane in an image sequence, it is harder to estimate the *fundamental matrix* F. The relation between F and $\mathcal{P}$ constrains F to be a member of a two parameter family effectively decoupling the rotational and translational components of F. $\mathcal{P}$ encodes the rotational components. This paper presents a series of methods for estimating F that make use of this property.

A test for 'compatibility' of rigid plane motions is derived based on the projected motion of the line of intersection of two planes.
Keywords: fundamental matrix, plane projectivity, epipolar geometry.

1. Introduction

The *fundamental matrix* is a key concept in vision using uncalibrated imagery. It encodes the epipolar geometry associated with the camera motion [7, 3, 6]. This may be used for motion segmentation [12, 8, 11, 10] or as the basis for recovering projective structure [6, 1, 2]. The projectivity of a moving plane also encodes information about epipolar geometry only less directly as it is also a function of scene structure [4, 9]. The fundamental matrix has proven complicated to estimate [7] because enforcing the $\det[\mathrm{F}] = 0$ constraint means that a set of non-linear equations have to be solved. If the set of points being viewed by the camera lie on a plane in 3D space then the equation used to estimate F is degenerate. The routines used to estimate F may be written to test the data for degeneracy [12] or planes explicitly tested for [9].

The relation between F and a plane projectivity $\mathcal{P}$ allows the representation of F to be broken into a rotational and a translational part. This gives rise to a new family of algorithms for estimating F. Given the existence of a plane in the field of view linear algorithms for estimating F that still enforce the det[F]=0 constraint are possible.

Given two distinct co-moving planar regions two new methods are presented for epipole estimation. One method uses linear least squares and hence the eigenvalues of the fit may be used a test for mutually compatible motion between planes, the second method is nonlinear.

The paper is layed out as follows. **Section 2** gives the notation used. **Section 3** gives the relation between the *fundamental matrix* and a plane projectivity and lists a series of algorithms for epipole estimation given $\mathcal{P}$. A comparison of results for all the various methods of F estimation is given in **section 4**. **Section 5** derives and demonstrates two tests for compatibility of plane motions. Conclusions are given in **section 6**

2. Notation

In this section all the notation used in this paper is set out. The camera model is a **pin hole** and for the purpose of this analysis need not be calibrated. The image plane is assumed to be in front of the center of projection.

- $\mathbf{x}_i$ the position of the i^{th} corner in homogeneous coordinates (by default in image 1)
- $\mathbf{x}'_i$ the matched position of the i^{th} corner in homogeneous coordinates (by default in image 2)
- $\mathcal{P}_{jk}$ the projectivity describing the transformation of a plane between image j to image k.
- F the fundamental matrix (by default from image 1 to image 2).
- $\mathbf{e}_{jk}$ the 'epipole' or projected position of camera center for image k in image j.

3. The relation between $\mathcal{P}$ and F

A complete exposition of the properties of the fundamental matrix and its relation to the essential matrix is given in [7]. If $\mathbf{x}$ and $\mathbf{x}'$ are the two image positions of a point under camera motion then the fundamental matrix F satisfies the following equation,

$$\mathbf{x}'^T \mathrm{F}\mathbf{x} = 0, \quad (1)$$

with the constraint

$$\det[\mathrm{F}] = 0. \quad (2)$$

If all of the points $\mathbf{x}$ lie on a plane in space then equation (1) is degenerate. However noting that the motion of points on the plane may be described by a projectivity $\mathcal{P}$,

$$\mathbf{x}' = \mathcal{P}\mathbf{x} \quad (3)$$

and substituting (3) into (1) gives [5],

$$\mathbf{x}^T \mathcal{P}^T F \mathbf{x} = 0 \quad (4)$$

for all $\mathbf{x}$. This implies that $\mathcal{P}^T F$ is antisymmetric. This is equivalent to the condition

$$F_{12} = [\mathbf{e}_{21}]_\times \mathcal{P} \tag{5}$$

where

$$[\mathbf{a}]_\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}.$$

$\mathcal{P}$ then encodes all the rotational information in F. Next a list of methods for estimating F or equivalently the epipole $\mathbf{e}_{jk}$ once $\mathcal{P}$ is known are given in terms of the function to be minimised..

1. Linear least squares,

   $$\min_F \sum_i \left(\mathbf{x}'^T F \mathbf{x}\right)^2.$$

   The problem becomes nonlinear if F is parameterised to enforce the condition det[F]=0.

2. Nonlinear least squares minimising the distance between image points and epipolar lines,

   $$\min_F \sum_i \left(\frac{1}{(F\mathbf{x}_i)_1^2 + (F\mathbf{x}_i)_2^2} + \frac{1}{(F\mathbf{x}'_i)_1^2 + (F\mathbf{x}'_i)_2^2}\right)\left(\mathbf{x}'^T F \mathbf{x}\right)^2.$$

   The condition det[F]=0 may be enforced through a particular parameterisation of F or as a side condition in the minimisation [7].

3. Parallax relative to a plane to directly estimate the epipole $\mathbf{e}$,

   $$\min_{\mathbf{e}_{21}} \sum_i \mathbf{e}_{21}^T \left[\frac{\mathcal{P}\mathbf{x}_i \times \mathbf{x}'_i}{|\mathcal{P}\mathbf{x}_i|\,|\mathbf{x}'_i|}\left(\frac{\mathcal{P}\mathbf{x}_i \times \mathbf{x}'_i}{|\mathcal{P}\mathbf{x}_i|\,|\mathbf{x}'_i|}\right)^T\right]\mathbf{e}_{21}$$

4. Linear least squares estimate for $\mathbf{e}$,

   $$\min_{\mathbf{e}_{21}} \sum_i \left(\mathbf{x}'^T [\mathbf{e}_{21}]_\times \mathcal{P}\mathbf{x}\right)^2.$$

   This may be solved directly as a minimum eigenvalue problem for the elements of the epipole $\mathbf{e}_{21}$. F is then found using (5) which naturally satisfies det[F]=0.

5. Nonlinear least squares minimising the distance between image points and epipolar lines for $\mathbf{e}$,

   $$\min_{\mathbf{e}_{21}} \sum_i \left(\frac{1}{([\mathbf{e}_{21}]_\times \mathcal{P}\mathbf{x}_i)_1^2 + ([\mathbf{e}_{21}]_\times \mathcal{P}\mathbf{x}_i)_2^2} + \frac{1}{([\mathbf{e}_{21}]_\times \mathcal{P}\mathbf{x}'_i)_1^2 + ([\mathbf{e}_{21}]_\times \mathcal{P}\mathbf{x}'_i)_2^2}\right)\left(\mathbf{x}'^T [\mathbf{e}_{21}]_\times \mathcal{P}\mathbf{x}\right)^2.$$

6. Three parameter linear least squares solution given two plane projectivities (simply using the antisymmetric property of $\mathcal{P}^T F$)

$$\min_{\mathbf{a}} \left| \left[\mathcal{P}_2^T \mathcal{P}_1^{-1} [\mathbf{a}]_\times\right] + \left[\mathcal{P}_2^T \mathcal{P}_1^{-1} [\mathbf{a}]_\times\right]^T \right|,$$

Where $|\ |$ represents the Frobenius norm of a matrix and $\left|[\mathbf{a}]_\times\right| = 1$, F then is given by $F = \mathcal{P}_1^{-T} [\mathbf{a}]_\times$ from (4).

7. Nonlinear method given two plane projectivities (derived using (5)).

$$\min_{\mathbf{e}_{21}} \sum_i 1 - \frac{\left|\left[[\mathbf{e}_{21}]_\times \mathcal{P}_1\right] \cdot \left[[\mathbf{e}_{21}]_\times \mathcal{P}_2\right]\right|}{\left|\left[[\mathbf{e}_{21}]_\times \mathcal{P}_1\right]\right| \left|\left[[\mathbf{e}_{21}]_\times \mathcal{P}_2\right]\right|}$$

8. The unique nondegenerate mutual eigenvector of two plane projectivities.

$$\mathcal{P}_1 \mathbf{e}_{12} = \lambda \mathcal{P}_2 \mathbf{e}_{12}$$

or

$$\mathcal{P}_2^{-1} \mathcal{P}_1 \mathbf{e}_{12} = \lambda \mathbf{e}_{12} \tag{6}$$

4. A comparison of method for F estimation

Results are presented for two directions of camera motion relative to the objects shown in figure 1. Table 1 gives results for the camera undergoing strict translation towards the object for each of the methods of epipole estimation given in section 3. In each case epipoles were computed between frame 1 and frame n for $n = 2...6$, the variance in angle with respect to epipole 1,6 and epipole 1,6 itself are given. Table 2 gives results for strict translation roughly parallel to the image array of the camera.

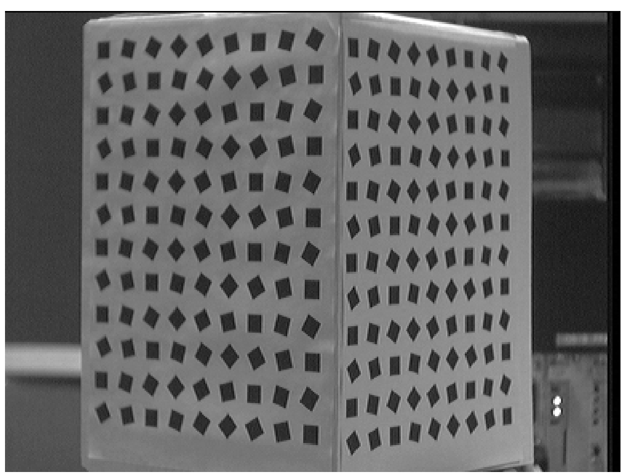

Figure 1: *Test object designed to provide stable corners for tracking.*

Method 3 has an encouragingly low variance indicating consistency but its estimates of the epipole are clearly biased. The epipoles estimated using method 7 have proven most reliable in a delicate reconstruction task.

Algorithm No.	epipole			variance
1	-0.08645	0.0533	0.9948	7.29
2	-0.07614	0.06811	0.9947	0.8432
3	-0.0866	0.06008	0.99442	0.3571
4	-0.08667	0.0600	0.9944	0.3567
5	-0.07903	0.05713	0.9952	0.7585
6	-0.0823	0.05888	0.9948	0.2271
7	-0.0757	0.05768	0.9954	0.08968
8	-0.0808	0.0570	0.9950	0.2237

Table 1: *The camera under went strict translation towards the reference object shown in figure 1. Algorithm number, estimate of the epipole, variance in the estimate computed over 5 frames.*

Algorithm No.	epipole			angular variance
1	0.9966	0.0645	-0.0508	20.12
2	0.9998	-0.0149	0.0101	21.83
3	0.9273	0.00635	-0.3740	2.86
4	0.9985	-0.0274	0.0472	1.247
5	0.9941	0.2488	0.2264	1237.68
6	0.9881	0.0136	-0.1535	21.34
7	0.9963	0.0108	-0.0845	12.97
8	0.9871	0.0113	-0.1598	20.87

Table 2: *The camera under went strict translation perpendicular to the reference object shown in figure 1. Algorithm number, estimate of the epipole, variance in the estimate computed over 5 frames.*

5. Testing two planes for compatible motion

It is desirable to be able to test whether two planes are moving in a rigidly connected manner. Two methods for testing the compatibility of plane motions are given here.

The first simply looks at the condition number of for the fit of the parameter **a** in algorithm number (6). Figure 2 shows a two link robot arm with 4 visible planes. The condition numbers for the parameter **a** are shown table 3. Planes a and b are co-moving, plane c and d are co-moving but independently of a and b. Unfortunately as can be seen from the table this test is inconclusive.

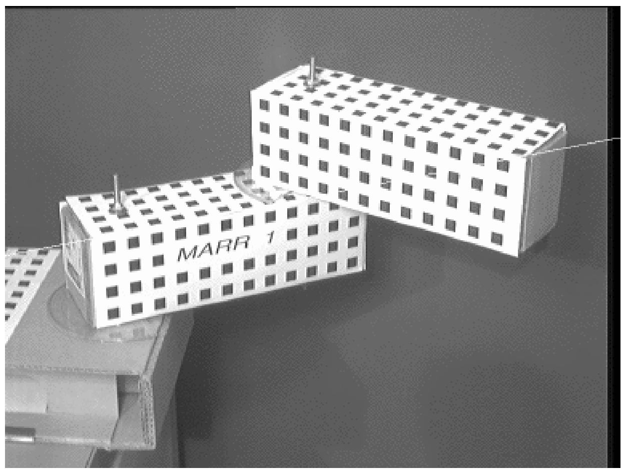

Figure 2: *A two link* Model Articulated Revolute Robot *generating two independent rigid motions. Link 1 is attached to the base and link 2 is attached to link 1. Plane a will be taken to be on to of link 2 plane b to be on the side of link 2 plane c on top of link 1 and plane d on the side of link 1. The estimated line of intersection between planes c and d is shown.*

plane 1	plane 2	1/(condition number)
a	b	0.0006533
a	c	0.0583333
a	d	0.2400
b	c	0.246399
b	d	0.001139
c	d	0.00846635

Table 3: *Table of the relative compatibility of pairs of planes in motion. Plane a is co-moving with plane b and plane c is co-moving with plane d (figure 2).*

The second test for plane compatibility functions by considering the projected motions of lines on a plane. The new position of a line $\mathbf{l}$ on a plane is given by,

$$\mathbf{l}' = \lambda \mathcal{P}^{-T} \mathbf{l}. \qquad (7)$$

The line of intersection of two planes in an image will then be an eigenvector of

$$\mathcal{P}_1^{-T}\mathbf{l} = \lambda \mathcal{P}_2^{-T}\mathbf{l}. \tag{8}$$

or

$$\left[\mathcal{P}_1^{-1}\mathcal{P}_2^{-T}\right]^T \mathbf{l}_i = \frac{1}{\lambda_i}\mathbf{l}_i. \tag{9}$$

This is very similar to (6) only with the matrix transposed and the reciprocal of the eigenvalue being taken. Matrices have the property that the eigenvalues of a matrix and its transpose are equal. Generally at most two of the $\mathbf{l}_i$ correspond to lines that project into the field of view. Empirically the eigen-systems of (6) and (9) have been found to have a curious property,

$$\mathbf{l}_i \perp \mathbf{x}_j \tag{10}$$

if $i \neq j$. The test for compatible plane motions is then to choose images directions $\mathbf{x}_{l_i}$ that span the projection of lines $\mathbf{l}_i$ into the second image and look at the angle between $\mathcal{P}_1\mathbf{x}_{l_i}$ and $\mathcal{P}_2\mathbf{x}_{l_i}$. If the angle is found to be equivalent to less than one pixel then you have found a preserved line of intersection of two planes and hence the two planes probably have one common rigid motion. If no preserved line of intersection exists then the planes are undergoing independent motion. A line of intersection recovered in this way is shown in figure 2.

6. Conclusions

A series of new methods for fundamental matrix estimation have been presented and compared with existing ones. Methods which base their estimate of F in part on a projectivity seem to give rise to estimates of epipoles with a lower variance. We also demonstrate that it is possible to determine whether planes are undergoing rigid co-motion. If two distinct co-moving planes have been found then the epipoles be estimated directly.

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