CSE276C - Differential Geometry

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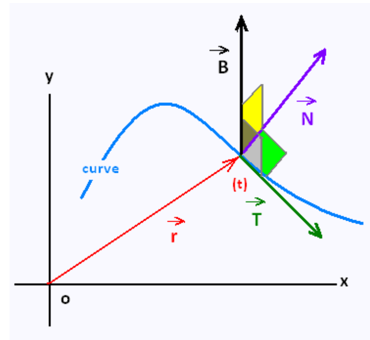
Introduction

- We can only touch on the basics, but valuable to have basic knowledge
- Differential Geometry is all about moving on a curve / manifold
- Robotics is all about moving considering not only kinematics, but also dynamics
- What motion is possible in a particular space

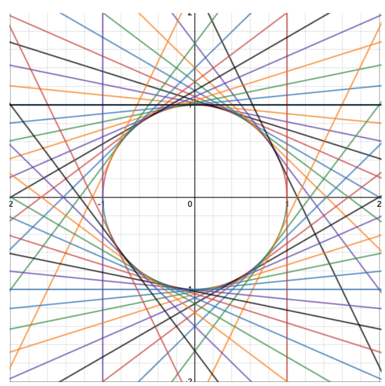
Basic Concepts

- Tangent vector
  - A vector anchored at a point p
  - Set of possible vectors for p is termed tangent space T_p



Basic Concepts

- Tangent Bundle
  - A space along with its tangent vectors
  - If $\mathbb{R}^n$ the underlying space and we have a tangent space of $\mathbb{R}^n$ anchored at each of the relevant points
  - Space is then $\mathbb{R}^n \times \mathbb{R}^n$
  - So a tangent bundle for a circle would be $S^1 \times \mathbb{R}^1$



Basic Concepts

- Vector Field
  - A function that maps a manifold to a tangent space
  - $M \rightarrow T(M)$ and within it $p \rightarrow v_p \in T_p$
  - Frequently denoted $V(p)$ or V_p
  - A classic question: does a manifold has a continuously changing vector field that is non-zero?

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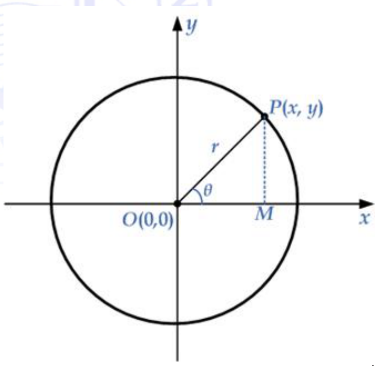
  - A classic question: does a manifold has a continuously changing vector field that is non-zero?
  - The circle example with $M = S^1$ is one such vector field

Geometry of curves in $\mathbb{R}^3$

- Consider parameterized curves $\alpha(t) = (x(t), y(t), z(t))$
- In general a curve α is a mapping $\alpha : I \to \mathbb{R}^3$
- I is an interval in $\mathbb{R}$ sometimes we will write it as $(\alpha_1(t), \alpha_2(t), \alpha_3(t))$
- In general $(x(t), y(t), z(t))$ are differentiable
- I.e., has derivatives of all orders throughout I

A simple 2D example

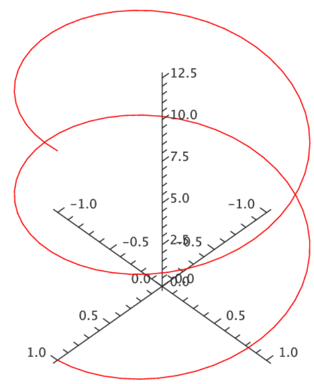


- $\alpha_1(\theta) = (r\cos(\theta), r\sin(\theta))$
- $\theta \in [0, 2\pi] = I$ OR
- $\alpha_2(\theta) = (r\cos(2\theta), r\sin(2\theta))$
- $\theta \in [0, \pi] = I$

Different curves / parameterizations can have the same trace

Simple 3D curve

- $\alpha(t) = (a\cos(t), a\sin(t), bt)$, with $t \in \mathbb{R}$



Velocity vector & Arclength

- The velocity vector of α at time t is the tangent vector of $\mathbb{R}^3$ given by

$$\alpha'(t) = (\alpha_1'(t), \alpha_2'(t), \alpha_3'(t))$$

- This vector is obviously also the tangent
- The speed of α is $v(t) = ||\alpha'(t)||$
- The arclength traversed between t_0 and t_1 is

$$\int_{t_0}^{t_1} v(t)dt$$

- You can re-parameterize $\alpha(t)$ as $\beta(s)$ where s is the arc-length, which is the same as representing α at unit speed

Simple Example – Helix

- Consider the helix: $\alpha(t) = (r\cos(t), r\sin(t), qt)$ then
  - Velocity: $\alpha'(t) = (-r\sin(t), r\cos(t), q)$
  - Speed: $v(t) = \sqrt{r^2 + q^2} = c$ a constant
  - Arc-length: $s(t) = \int_0^t c dt = ct$. Thus $t(s) = \frac{s}{c}$
  - Re-parameterized: $\beta(s) = \alpha(\frac{s}{c}) = (r\cos(\frac{s}{c}), r\sin(\frac{s}{c}), q\frac{s}{c})$

Arclength?

- So does the integral

$$s(t) = \int_{t_0}^{t_1} ||\alpha'(t)|| dt$$

always converge?

Arclength?

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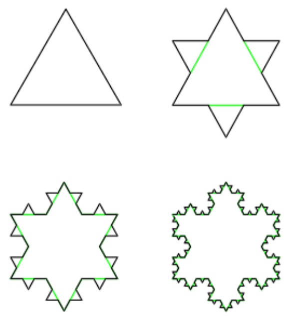
$$s(t) = \int_{t_0}^{t_1} ||\alpha'(t)||dt$$

always converge?
- Some curves have infinite arclength (ex fractals - Koch Snowflake)



Vector fields of β

- We can define a set of vector fields for β
  - $T = \beta'$ the unit tangent field
  - $N = \frac{T'}{||T'||}$ the principal normal vector field
  - $B = T \times N$ called the bi-normal vector field of β
- The quantity $||T'||$ is also named the curvature function $K(s) = ||T'(s)||$
- The triple (T,N,B) is called the Frenet Frame field of β

Curvature

- Let $\alpha : I \rightarrow \mathbb{R}^3$ be a curve parameterized by arclength
- Curvature is then defined as $||\alpha''(s)|| = K(s)$
- $\alpha'(s)$ – the tangent vector of s
- $\alpha''(s)$ – the change in the tangent vector
- $R(s) = 1/K(s)$ – is called the radius of curvature

Simple examples

- Straight line

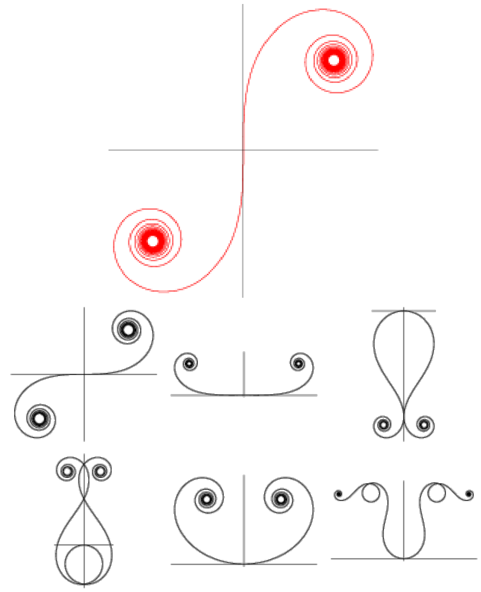
$$\begin{aligned} \alpha(s) &= us + v, \;\; u, v \in \mathbb{R}^2 \\ \alpha'(s) &= u \\ \alpha''(s) &= 0 \Rightarrow ||\alpha''(s)|| = 0 \end{aligned}$$

- Circle

$$\begin{aligned} \alpha(s) &= (a\cos(s/a), a\sin(s/a)), \;\; s \in [0, 2\pi a] \\ \alpha'(s) &= (-\sin(s/a), \cos(s/a)) \\ \alpha''(s) &= (-\cos(s/a)/a, -\sin(s/a)/a) \Rightarrow ||\alpha''(s)|| = 1/a \end{aligned}$$

Curvature examples

- Cornu Spiral - K(s) = s
- Generalized Cornu Spirals - K(s) - Polynomial of s



Normals

- When α is parameterized by arc length

$$\alpha'(s) \cdot \alpha'(s) = 1$$

- From Vector Calculus
  - If f, g: $I \to \mathbb{R}^3$ and $f(t) \cdot g(t) = const$ for all t
  - then

$$f'(t) \cdot g(t) = -f(t) \cdot g'(t)$$

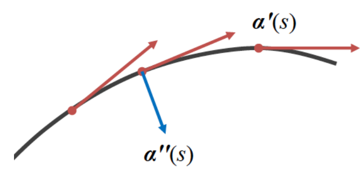
  for f * f this is only true for f'(t) f(t) = 0
- This implies that

$$\alpha''(s) \cdot \alpha'(s) = 0$$

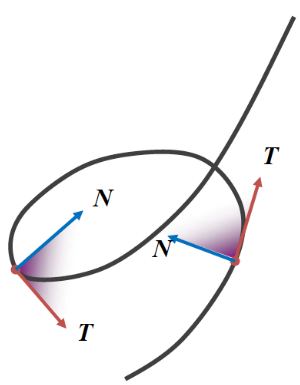
  or $\alpha''(s)$ is orthogonal to $\alpha'(s)$
- Its proportional to the normal of the curve

Normals

- $\alpha'(s) = T(s)$ – Tangent Vector
- $||\alpha'(s)||$ – arc length
- $\alpha''(s) = T'(s)$ – normal direction
- $||\alpha''(s)||$ – curvature
- If $||\alpha''(s) \neq 0$ then $\alpha''(s) = T'(s) = K(s)N(s)$



Osculating Plane



Source: M. Ben-Chen, Stanford

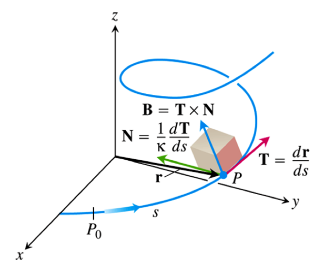
- The local plane determined by the unit tangent and the normal vectors - T(s) and N(s) is call the osculating plane at s

The Bi-normal Vector

- The binormal is defined for $K(s) \neq 0$ by

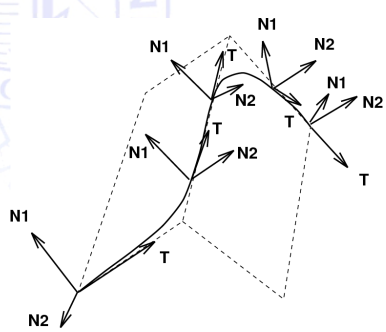
$$B(s) = T(s) \times N(s)$$

- The bi-normal defines the osculating plane



Source: R. Gardner, ETSU

The Frenet Frame



Source: A. J. Hanson, LBL

- The system $\{T(s), N(s), B(s)\}$ for an ortho-normal basis for $\mathbb{R}^3$ called the Fernet Frame
- The obvious question - How does it change along a curve? I.e., what are T'(s), N'(s), and B'(s)?

T'(s)

- We have already covered T'(s)

$$T'(s) = K(s)N(s)$$

- As it is in the direction of N(s) it is orthogonal to B(s) and T(s).

N'(s)

- We know that $N(s) \cdot N(s) = 1$
- From our earlier lemma (vector calculus) $N'(s) \cdot N(s) = 0$
- We know $N(s) \cdot T(s) = 0$ from the lemma $N'(s) \cdot T(s) = -N(s) \cdot T'(s)$
- Given $K(s) = N(s) \cdot T'(s)$
- It must be true that $N'(s) \cdot T(s) = -K(s)$

Torsion

- For the parameterized curve $\alpha : I \rightarrow \mathbb{R}^3$ the torsion of α is defined by

$$\tau(s) = N'(s) \cdot B(s)$$

- We can then express

$$N'(s) = K(s)T(s) + \tau(s)B(s)$$

Curvature vs Torsion

- **Curvature** indicates how much the normal changes in the direction of the tangent
- **Torsion** indicates how much the normal change in the direction orthogonal to the osculating plane
- Curvature is always positive, the torsion can be negative
- Neither depend on the choice of parameterization

B'(s)

- We know that $B(s) \cdot B(s) = 1$
- From the lemma we know $B'(s) \cdot B(s) = 0$
- We further know: $B(s) \cdot T(s) = 0$ and $B(s) \cdot N(s) = 0$
- From the lemma:

$$B'(s) \cdot T(s) = -B(s) \cdot T'(s) = B(s) \cdot K(s)N(s) = 0$$

- We get

$$B'(s) \cdot N(s) = -B(s) \cdot N'(s) = -\tau(s)$$

  and from this we have

$$B'(s) = -\tau(s)N(s)$$

The Frenet Formulas

$$\begin{array}{rclll} T'(s) & = & & K(s)N(s) & \\ N'(s) & = & -K(s)T(s) & & +\tau(s)B(s) \\ B'(s) & = & & -\tau(s)N(s) & \end{array}$$

In Matrix Form

$$\begin{pmatrix} | & | & | \\ T'(s) & N'(s) & B'(s) \\ | & | & | \end{pmatrix} = \begin{pmatrix} | & | & | \\ T(s) & N(s) & B(s) \\ | & | & | \end{pmatrix} \begin{pmatrix} 0 & K(s) & 0 \\ K(s) & 0 & -\tau(s) \\ 0 & \tau(s) & 0 \end{pmatrix}$$

Example - Back to the helix

- For: $\alpha(t) = (a\cos(t), a\sin(t), bt)$
- Re-parameterized: $\alpha(s) = (a\cos(s/c), a\sin(s/c), bs/c)$ where $c = \sqrt{a^2 + b^2}$
- Curvature is then: $K(s) = \frac{a}{a^2+b^2}$
- Torsion is then $\tau(s) = \frac{}{a^2+b^2}$
- Note for this example both curvature and torsion are constants

Covariant Derivatives and Lie Brackets

- Suppose $V \& W$ are two vector fields in $\mathbb{R}^n$ so that for each point $p \in \mathbb{R}^n$ $V(p)$ and $W(p)$ are vectors in $\mathbb{R}^n$
- The **covariant derivative** of W wrt V is

$$(\nabla_v W)(p) = \frac{d}{dt} W(p + tV_p)|_{t=0}$$

- $\nabla_v W$ measures the change in W as one moves along V

Examples - covariant derivatives

- In $\mathbb{R}^2$ W(p) = (1,0) and V(p) = (0,1) forall p
- The $\nabla_v W = \nabla_w V = 0$
- For a circle in 2D, $p = (x, y) \in \mathbb{R}^2$

$$W = \frac{(x, y)}{\sqrt{x^2 + y^2}} \text{ and } V = \frac{(-y, x)}{\sqrt{x^2 + y^2}}$$

- Then $\nabla_v W = \frac{v}{\sqrt{x^2+y^2}}$ and of course $\nabla_w V = 0$

A few things about covariant derivatives

- $\nabla_v W$ is an n-dimensional vector
- $\nabla_v(aW + bU) = a\nabla_v W + b\nabla_v U$
- $\nabla_{fV+gU} W = f\nabla_v W + g\nabla_u W$

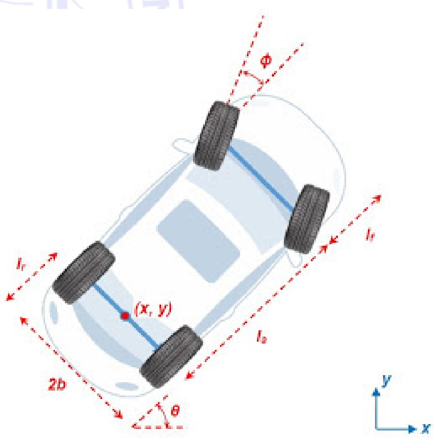
Lie Bracket

- The **Lie Bracket** [V, W] of the two vector fields is defined to be

$$[V, W] = \nabla_V W - \nabla_W V$$

- Basically measure flow in the directions of V, -V, W, -W
- Lets illustrate this with a real robot example

Parallel Parking



- The configuration - (x, y, θ)
- The controls are (v, ϕ)
- The controls are

$$\begin{aligned}\dot{x} &= v\cos\phi\cos\theta \\ \dot{y} &= v\cos\phi\sin\theta \\ \dot{\theta} &= \frac{v}{l}\sin\phi\end{aligned}$$

- We can consider nominal motion $(1, \phi_1)$ and $(1, \phi_2)$ as wheel directions

Parallel Parking - Cont

- Two vector fields

$$V_i = V_i(x, y, \theta) = (\cos\phi_i \cos\theta, \cos\phi_i \sin\theta, \frac{\sin\phi_i}{l})$$

- Then

$$\nabla_{V_1} V_2 = (\nabla(\cos\phi_1 \cos\theta)V_2, \nabla(\cos\phi_1 \sin\theta)V_2, \nabla(\frac{\sin\phi_1}{l})V2)$$

skipping calculations

$$\nabla_{V_1} V2 = \frac{\sin\phi_1 \cos\phi_2}{l}(-\sin\theta, \cos\theta, 0)$$

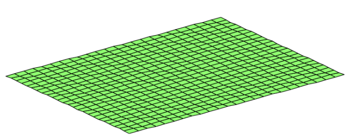
and similarly for the

$$[V_1, V_2] = \frac{\sin(\phi_1 - \phi_2)}{l}(-\sin\theta, \cos\theta, 0)$$

So we can move perpendicular to the axis as long as $(\phi_1 - \phi_2) \neq 0$

Moving to manifolds

- Smooth Manifolds
  - A manifold is a set M with an associated one-to-one map $\phi : U \to M$ from an open subset $U \subset \mathbb{R}^m$ called a global chart or coordinate system of M



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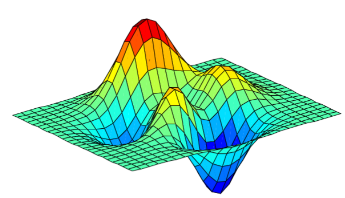


Smooth Manifolds

- A smooth manifold is a pair $(M, \mathcal{A})$ where:
  - M is a set
  - $\mathcal{A}$ is a family of 1-1 charts: $\phi : U \to M$ from some open subset $U = U_\phi \subset \mathbb{R}^m$ for M

Differentiable and smooth functions

- $f : U \subset \mathbb{R}^n \to \mathbb{R}^q$

$$(y_1, \ldots, y_q) = f(x_1, \ldots, x_n)$$

- f is of a class C^r if f has continuous partial derivatives

$$\frac{\partial^{r_1+\ldots+r_n} y_k}{\partial x_1^{r_1} \ldots \partial x_n^{r_n}}$$

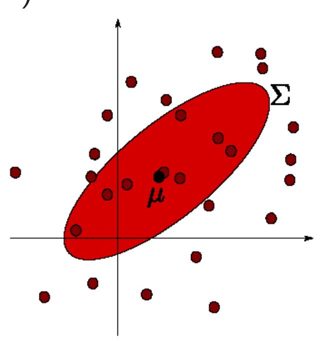
- If $r = \infty$, then f is **smooth**, the main focus in robotics

Diffeomorphism

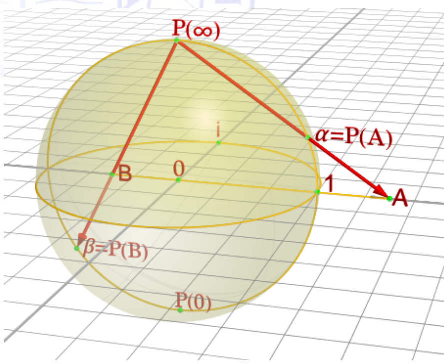
- When n = q
  - if f is 1-1, f and f^{-1} are both C^r
  - $\Rightarrow$ f is a **C^r-diffeomorphism**
  - Smooth diffemorphisms are simply referred as diffeomorphisms
- Inverse Function Theorem:
  - f diffeomorphism $\Rightarrow det(J_x f) \neq 0$
  - $det(J_x f) \neq 0 \Rightarrow f$ is local diffeomorphism in a neighborhood of x

Example - Gaussian Distribution

- The space of n-dimensional Gaussian distributions is a smooth manifold
- Global chart: $(\mu, \Sigma) \in \mathbb{R}^n \times \mathcal{P}(n)$



Manifolds can generate multiple charts



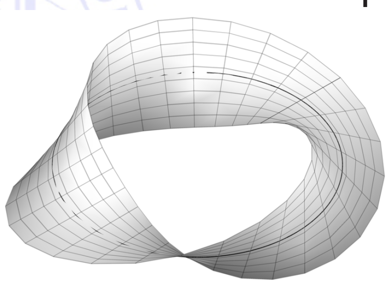
- The sphere $\mathcal{S}^2 = \{(x, y, x), x^2 + y^2 + z^2 = 1\}$ has multiple projections/charts
- We can project from the North Pole, of a point P = (x,y,z) given by

$$\phi(P) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

- is a large coordinate system around the south pole

Manifolds requiring multiple chartss

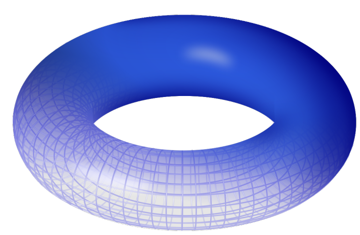
The Moebius Strip



$$u \in [0, 2\pi], v \in [-1/2, 1/2]$$

$$\begin{pmatrix} \cos(u)\left(1 + \frac{1}{2} v \cos\left(\frac{u}{2}\right)\right) \\ \sin(u)\left(1 + \frac{1}{2} v \cos\left(\frac{u}{2}\right)\right) \\ \frac{1}{2} v \sin\left(\frac{u}{2}\right) \end{pmatrix}$$

2D Torus



$$(u, v) \in [0, 2\pi]^2, R >> r > 0$$

$$\begin{pmatrix} \cos(u)\,(R + r\cos(v)) \\ \sin(u)\,(R + r\cos(v)) \\ r\sin(v) \end{pmatrix}$$

Summary

- Covering basics of movement along curves
- Many more derivations can be provided for movement on manifolds
- Covering basic characteristics of curves and manifolds
- Definition of the Frenet frame and associated characteristics
- Brief coverage of covariant derivatives and Lie bracket